An analysis of the coefficient of f^2 shows that it is negative for any lags of the condensed phase (any relative slips $\rho_1/\rho_2 > 3\varkappa/(\varkappa+1)$ ($\beta < 0.419$, in the case $\varkappa = 5/4$). Here the singularity realized in a divergent channel is obviously a saddle-point if we do not bear in mind drop production nor the influence of the curvature of the channel profile. Drop agglomeration ($\Phi < 0$), which predominates over fragmentation, and the positive curvature of the channel profile ($y'' > 0$) only strengthen this conclusion. The saddle-point nature of the singularity at higher contents of condensed phase predominating over drop fragmentation in the negative curvature of the profile (i.e., the same as in the case of a pure gas) is possible only for slips of not too high a magnitude. In the opposite case $\det \| a_{ij} \| > 0$ and the nature of the singularity differs. The transition point beyond the speed of sound loses the nature of a saddle point.

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USE OF THE PARAMETRIX METHOD FOR ESTIMATING EFFECTIVE ELASTIC MODULI OR RANDOMLY NONHOMOGENEOUS ELASTIC BODIES

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UDC 539.31

The magnitude of the elasticity tensor of a comparison body remains unclarified if we use a singular approximation [1] to estimate the effective values of the elasticity tensor. Below we will use a parametrix method [2] to determine the first approximation of the random component of the deformation tensor and the effective values of the elasticity tensor, and will also compare the exact solution for one particular heterogeneous and a previously used approximation.

The effective value of the elasticity tensor $\boldsymbol{\lambda}^0$ is determined by

$$\boldsymbol{\lambda}^0 \langle \boldsymbol{\varepsilon} \rangle = \langle \boldsymbol{\lambda} \rangle \langle \boldsymbol{\varepsilon} \rangle + \langle \boldsymbol{\lambda}' \boldsymbol{\varepsilon}' \rangle,$$

where $\boldsymbol{\lambda}' = \boldsymbol{\lambda} - \langle \lambda \rangle$; $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon} - \langle \boldsymbol{\varepsilon} \rangle$, and the stress tensor satisfies the equilibrium equation

$$\nabla(\boldsymbol{\lambda}\boldsymbol{\varepsilon}) = 0.$$

The solution of the equation will be found in the form of a space potential

$$\boldsymbol{\varepsilon}'' = \int \mathrm{def}_x \mathbf{G}(\mathbf{x}, \mathbf{y}) \mathbf{f}(\mathbf{y})\, dv_y, \tag{1}$$

where $\mathrm{def}_x = (1/2)[\nabla x + (\nabla x)^T]$; and $\mathbf{G}(\mathbf{x}, \mathbf{y})$ is the parametrix [4] of the equilibrium equation, which coincides with the "principal" polar part of Green's tensor of a heterogeneous and isotropic medium.

We assume that $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0 + \boldsymbol{\varepsilon}''$, $\boldsymbol{\varepsilon}^0 = \mathrm{const}$, and substituting Eq. (1) in the equilibrium equation, we obtain the integral equation

Tallin. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 171-175, January-February, 1976. Original article submitted March 18, 1975.

$$f(x) = \nabla_x \lambda(x)\varepsilon^0 + \int \nabla_x \lambda(\mathbf{x}) \operatorname{def}_x \mathbf{G}(\mathbf{x}, \mathbf{y}) \mathbf{f}(\mathbf{y})\, dv_y, \tag{2}$$

whose solution is constructed as usual by the iteration method.

Let us consider in detail the case of an isotropic, randomly heterogeneous medium

$$\lambda_{ijmn} = \lambda\delta_{ij}\delta_{mn} + 2\mu\delta_{im}\delta_{jn}.$$

The parametrix of the equilibrium equation [2] is given by

$$G_{ij}(\mathbf{x}, \mathbf{y}) = \frac{1}{8\pi[\lambda(\mathbf{y}) + 2\mu(\mathbf{y})]} r_{,ij} + \frac{1}{8\pi\mu(\mathbf{y})}(\delta_{ij} r_{,pp} - r_{,ij}),$$

where $r = |\mathbf{x}-\mathbf{y}|$; and δ_{ij} is the Kronecker symbol.

We will limit ourselves to the first approximation in order to estimate the random component of the deformation tensor. In this case

$$\langle \lambda' u''_{i,i} \rangle = \frac{1}{8\pi} \int \frac{r_{,iji}}{r} \left[H_1(r) r_j \varepsilon^0_{nn} + 2H_2(r) r_m \varepsilon^0_{mj}\right] dv_y; \tag{3}$$

$$\langle \mu' u''_{i,k} \rangle = \frac{1}{8\pi} \int \frac{r_{,ijk}}{r} \{[H_3(r) + H_5(r)] r_j \varepsilon^0_{nn} + 2[H_4(r) - H_6(r)]$$
$$\times r_m \varepsilon^0_{mj} + [H_5(r) r_j \varepsilon^0_{nn} + 2H_6(r) r_m \varepsilon^0_{mj}]\} \delta_{ij} r_{,nnk} dv_y,$$

where we have set [5]

$$\left\langle \frac{\lambda'(\mathbf{x})\nabla\lambda(\mathbf{y})}{\lambda(\mathbf{y}) + 2\mu(\mathbf{y})} \right\rangle = H_1 \frac{\mathbf{r}}{r}; \quad \left\langle \frac{\lambda'(\mathbf{x})\nabla\mu(\mathbf{y})}{\lambda(\mathbf{y}) + 2\mu(\mathbf{y})} \right\rangle = H_2(r)\frac{\mathbf{r}}{r};$$
$$\left\langle \frac{\mu'(\mathbf{x})\nabla\lambda(\mathbf{y})}{\lambda(\mathbf{y}) + 2\mu(\mathbf{y})} \right\rangle = H_3(r)\frac{\mathbf{r}}{r}; \quad \left\langle \frac{\mu'(\mathbf{x})\nabla\mu(\mathbf{y})}{\lambda(\mathbf{y}) + 2\mu(\mathbf{y})} \right\rangle = H_4(r)\frac{\mathbf{r}}{r};$$
$$\left\langle \frac{\mu'(\mathbf{x})\nabla\lambda(\mathbf{y})}{\mu(\mathbf{y})} \right\rangle = H_5(r)\frac{\mathbf{r}}{r}; \quad \left\langle \frac{\mu'(\mathbf{x})\nabla\mu(\mathbf{y})}{\mu(\mathbf{y})} \right\rangle = H_6(r)\frac{\mathbf{r}}{r}.$$

for the cross-correlation functions for the case of randomly isotropic uniform fields.

Transforming the integrals (3), we obtain an estimate of the effective values,

$$\mu_1 = \langle\mu\rangle - \frac{6}{15}\langle\mu' \ln\mu\rangle + \frac{4}{15}\int_0^\infty H_4(r)\,dr; \tag{4}$$

$$\lambda_1 = \langle\lambda\rangle + \int_0^\infty \left\{H_1(r) + \frac{2}{3}[H_2(r) + H_3(r)] + \frac{4}{15}[H_4(r) - H_6(r)]\right\} dr.$$

Here, we have used the equations

$$\frac{1}{4\pi}\int H(r)\frac{r_k r_m}{r^4}\,dv = \frac{1}{3}\int_0^\infty H(r)\,dr\,\delta_{km};$$

$$\frac{1}{4\pi}\int H(r)\frac{r_k r_m r_i r_j}{r^6}\,dv = \frac{1}{15}\int_0^\infty H(r)\,dr\,[\delta_{mk}\delta_{ij} + \delta_{ki}\delta_{mj} + \delta_{kj}\delta_{im}].$$

We may assume that the resulting equations have a relatively high precision for a random medium with constant Poisson coefficient, i.e., $\mu(\mathbf{r}) = k\lambda(\mathbf{r})$, since in this case the parametrix is the influence function of the equation $\lambda_{ijmn}\nabla_j\varepsilon_{mn} = 0$. The effective values correspondingly have the form

$$\mu_1 = \langle\mu\rangle - [(6 + 16k)/15(1 + 2k)]\langle\mu' \ln\mu\rangle,$$
$$\lambda_1 = \langle\lambda\rangle - [(15 + 12k - 4k^2)/15(1 + 2k)k]\langle\mu' \ln\mu\rangle.$$

The exact effective value λ_1 has been calculated [6, 7] for a heterogeneous two-phase medium with $\mu = \text{const}$. We will calculate in the general case for a medium with constant rigidity modulus. We take the divergence from the equilibrium equation obtaining

$$\operatorname{grad}[(\lambda + 2\mu)\operatorname{div}\mathbf{u}] - \mu \operatorname{rot}\operatorname{rot}\mathbf{u} = 0,$$

$\Delta[(\lambda + 2\mu)\operatorname{div}\mathbf{u}] = 0.$

According to the mean value theorem [8] for harmonic functions,

$$(\lambda + 2\mu)\operatorname{div}\mathbf{u} = \int(\lambda + 2\mu)\operatorname{div}\mathbf{u}\, ds/4\pi R = \langle(\lambda + 2\mu)\operatorname{div}\mathbf{u}\rangle. \tag{5}$$

Integration is conducted over a sphere of arbitrary radius, i.e., averaging relative to a spherical layer yields a value $(\lambda+2\mu)$ div **u** at the center of the sphere. We find from Eq. (5) that

$$\langle \mathrm{div}\ \mathbf{u}\rangle = (\langle \lambda\ \mathrm{div}\ \mathbf{u}\rangle + 2\mu\langle \mathrm{div}\ \mathbf{u}\rangle\langle 1/(\lambda+2\mu)\rangle,$$

and, consequently,

$$\lambda_1 = \langle \lambda\ \mathrm{div}\ \mathbf{u}\rangle/\langle \mathrm{div}\ \mathbf{u}\rangle = \langle 1/(\lambda+2\mu)\rangle^{-1} - 2\mu. \tag{6}$$

To compare the estimate (4) and a previous estimate [3] to the exact value, we will decompose (6) in a series in moments of the random component,

$$\lambda_1 = \frac{\langle \lambda+2\mu\rangle}{1+\sum\limits_{k=2}^{\infty}(-1)^k\frac{m_k}{\langle\lambda+2\mu\rangle^k}} - 2\mu = \langle\lambda\rangle - \frac{m_2}{\langle\lambda+2\mu\rangle} + \frac{m_2}{\langle\lambda+2\mu\rangle^2} - \ldots, \tag{7}$$

where

$$m_k = \langle(\lambda - \langle\lambda\rangle)^k\rangle = \langle(\lambda')^k\rangle.$$

The estimate (4) for this medium

$$\lambda_1 = \langle\lambda\rangle - \langle\lambda' \ln(\lambda+2\mu)\rangle = \langle\lambda\rangle - m_2/\langle\lambda+2\mu\rangle + (1/2)m_3/\langle\lambda+2\mu\rangle^2 - \ldots$$

is more accurate to within three terms than [3], which gave only the first two terms of the expansion (7).

The exact value (6) can be obtained by using Green's tensor for the given medium

$$G_{ji}(\mathbf{x},\mathbf{y}) = \frac{-1}{16\pi^2}\int \frac{1}{\lambda(\mathbf{z})+2\mu}\left(\frac{\partial}{\partial z_i}\frac{1}{|\mathbf{z}-\mathbf{y}|}\right)\left(\frac{\partial}{\partial x_j}\frac{1}{|\mathbf{z}-\mathbf{x}|}\right)dv_z + \frac{1}{8\pi\mu}(\delta_{ij}r_{,pp} - r_{,ji}). \tag{8}$$

Let us prove that when $\lambda(z)=$const, Eq. (8) is transformed into a well-known equation for a homogeneous medium, i.e., the integral

$$-\int\left(\frac{\partial}{\partial z_i}\frac{1}{|\mathbf{y}-\mathbf{z}|}\right)\left(\frac{\partial}{\partial x_j}\frac{1}{|\mathbf{z}-\mathbf{x}|}\right)dv_z = \frac{\partial}{\partial y_i}\int\frac{1}{|\mathbf{y}-\mathbf{z}|}\frac{\partial}{\partial x_j}\frac{1}{|\mathbf{x}-\mathbf{z}|}dv_z = 2\pi\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}r. \tag{9}$$

The integral in the right side of Eq. (9) is the derivative of the space potential (attraction) with density $1/\rho$ ($\rho=|\mathbf{y}-\mathbf{z}|$) for an infinite space and, according to [9], is equal to the integral over a sphere with center at y and radius $|\mathbf{y}-\mathbf{x}|=r$,

$$\int\limits_v \frac{1}{|\mathbf{y}-\mathbf{z}|}\frac{\partial}{\partial x_j}\frac{1}{|\mathbf{x}-\mathbf{z}|}dv_z = 4\pi\int\limits_0^r \rho d\rho\frac{\partial}{\partial x_j}\frac{1}{r} = -2\pi\frac{x_j-y_j}{r}.$$

Direct substitution of Eq. (8) in the equilibrium equation verifies that Eq. (8) is a Green's function.

We will take as the Levi function [4]

$$G^1_{ji}(\mathbf{x},\mathbf{y}) = \frac{-1}{16\pi^2}\int\frac{1}{\lambda(\mathbf{z})+2\mu(\mathbf{z})}\left(\frac{\partial}{\partial z_i}\frac{1}{|\mathbf{z}-\mathbf{y}|}\right)\left(\frac{\partial}{\partial x_j}\frac{1}{|\mathbf{z}-\mathbf{x}|}\right)dv_z + \frac{1}{8\pi\mu(\mathbf{y})}(\delta_{ij}r_{,pp} - r_{,ji}). \tag{10}$$

We will calculate ε''_{ij} using Eq. (2) to a first approximation, Eq. (10) being utilized as the integrand.

Introducing the cross-correlation function of isotropic uniform fields λ and μ,

$$\langle\nabla_y\lambda(y)/[\lambda(z)+2\mu(z)]\rangle = M_1(\rho)\boldsymbol{\rho}/\rho;\ \langle\nabla_y\mu(y)/[\lambda(z)+2\mu(z)]\rangle = M_2(\rho)\boldsymbol{\rho}/\rho;$$

$$\langle\lambda'(x)\nabla_y\lambda(y)/[\lambda(z)+2\mu(z)]\rangle = \nabla_y H_1(\rho, R, \theta)\rangle;\ \langle\lambda'(x)\nabla_y\mu(y)/[\lambda(z)+2\mu(z)]\rangle = \nabla_y H_2(\rho, R, \theta),$$

where $\rho=|\mathbf{y}-\mathbf{z}|$; $R=|\mathbf{x}-\mathbf{z}|$; and $\cos\theta = \boldsymbol{\rho}\mathbf{R}/\rho R$,

$$\langle\varepsilon''_{ik}\rangle = \frac{1}{3}\int\limits_0^\infty\left[M_1(\rho)+\frac{2}{3}M_2(\rho)\right]d\rho\,\varepsilon^0_{pp}\delta_{ik} = -\left(\frac{1}{3}\left\langle\frac{\lambda'}{\lambda+2\mu}\right\rangle + \frac{2}{3}\left\langle\frac{\mu'}{\lambda+2\mu}\right\rangle\right)\varepsilon^0_{pp}\delta_{ik}. \tag{11}$$

In calculating the integrals

$$\langle \lambda' \varepsilon''_{pp} \rangle = \frac{-1}{16\pi^2} \int \int \left\{ \frac{\partial}{\partial y_i} H_1(\rho, R, \theta)\, \varepsilon^0_{pp} + \frac{\partial}{\partial y_k} H_2\ (\rho, R, \theta)\, \varepsilon^0_{ki} \right\} \left(\frac{\partial}{\partial z_i} \frac{1}{\rho} \right) \left(\frac{\partial}{\partial x_p} \frac{\partial}{\partial x_p} \frac{1}{R} \right) dv_y dv_z$$
$$= -\left[\left\langle \frac{(\lambda')^2}{\lambda + 2\mu} \right\rangle + \frac{2}{3} \left\langle \frac{\lambda' \mu'}{\lambda + 2\mu} \right\rangle \right] \varepsilon^0_{pp}; \tag{12}$$

$$\langle \mu' \varepsilon''_{pp} \rangle = -\left(\left\langle \frac{\mu' \lambda'}{\lambda + 2\mu} \right\rangle + \frac{2}{3} \left\langle \frac{\mu' \mu'}{\lambda + 2\mu} \right\rangle \right) \varepsilon^0_{pp} \tag{13}$$

it is assumed that integration of y over the entire space results in an equation that depends on R, such that

$$\frac{1}{4\pi} \int H(R) \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} \frac{1}{R}\, dv_z = -\frac{1}{3} H(0)\, \delta_{ij}.$$

The estimate of the effective value of the compression modulus $k = \lambda + (2/3)\mu$ is given by

$$k_1 = \langle k \rangle + \left\langle \left(\lambda' + \frac{2}{3} \mu' \right) \varepsilon''_{pp} \right\rangle \frac{1}{\langle \varepsilon^0_{pp} + \varepsilon''_{pp} \rangle} = \langle k \rangle - \frac{\left\langle \frac{\lambda' \lambda'}{\lambda + 2\mu} \right\rangle + \frac{8}{9} \left\langle \frac{\lambda' \mu'}{\lambda + 2\mu} \right\rangle + \frac{4}{9} \left\langle \frac{\mu' \mu'}{\lambda + 2\mu} \right\rangle}{1 - \frac{1}{3} \left\langle \frac{\lambda'}{\lambda + 2\mu} \right\rangle - \frac{2}{9} \left\langle \frac{\mu'}{1 + 2\mu} \right\rangle}, \tag{14}$$

taking into account Eq. (11)-(13). When μ = const, Eq. (14) is transformed into the exact solution. We will assume that $1/(\lambda + 2\mu)$ is given by

$$1/(\lambda + 2\mu) \approx 1/\langle \lambda + 2\mu \rangle, \quad \text{or by} \quad 1/(\lambda + 2\mu) \approx \langle 1/(\lambda + 2\mu) \rangle,$$

in Eq. (14), obtaining estimates that correspond to the singular approximation [1] and also those obtained in [3].

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